Nassau County Interscholastic Mathematics League

Contest #3 Answers must be integers from 0 to 999, inclusive. 2021 – 2022

No calculators are allowed.

Time: 10 minutes **Name: _____**

13. Compute the smallest positive integer x that makes $\sqrt{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot x}$ an integer.

14. In an arithmetic sequence, the sum of the first 50 terms is 2225 and the sum of the next 50 terms is 4725. Compute the first term of the sequence.

13. ☐ 14. ☐

Nassau County Interscholastic Mathematics League

Contest #3 Answers must be integers from 0 to 999, inclusive. 2021 – 2022

No calculators are allowed.

Time: 10 minutes **Name: _____**

15. Compute the largest two-digit integer that is equal to three times the product of its digits.

16. In right ΔABC, point M is the midpoint of hypotenuse $\overline{AB}$. Point D is on leg $\overline{BC}$ and $\overline{DM} \perp \overline{AB}$. If $AC = 12$, $CD = 5$, and AB, in simplest radical form, is $p\sqrt{q}$, compute pq.

15. ☐ 16. ☐

Nassau County Interscholastic Mathematics League

Contest #3 Answers must be integers from 0 to 999, inclusive. 2021 – 2022

No calculators are allowed.

Time: 10 minutes **Name: _____**

17. Compute the product: 10 times the sum of the roots of $(2x-3)^3 = 2x-3$.

18. If $0 < x \leq \frac{\pi}{2}$ and $\csc x + \cot x = 3$, compute $144(\csc x - \cot x)$.

    Note: $(\csc x)^2 = (\cot x)^2 + 1$

17. ☐ 18. ☐

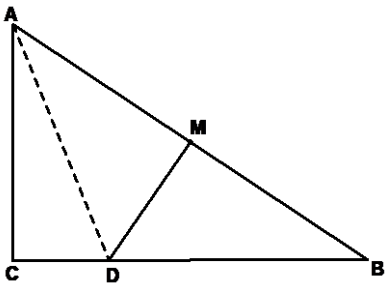
Solutions for Contest #3

13. The problem requires the radicand to be a perfect square. The radicand can be re-written as $3^2 \cdot 2^3 \cdot 7 \cdot 2 \cdot 3 \cdot 5 \cdot 2^2 \cdot 3 \cdot 2 \cdot x = 3^4 \cdot 2^7 \cdot 7 \cdot 5 \cdot x$. So, the minimum value of x that makes the radicand a perfect square is $2 \cdot 7 \cdot 5 = \mathbf{70}$.

14. Using the formula $S = \frac{n}{2}\big(2a_1 + d(n-1)\big)$, the sum of the first 50 terms is $\frac{50}{2}(2a_1 + 49d) = 50a_1 + 1225d = 2225$. The sum of all 100 terms is $\frac{100}{2}(2a_1 + 99d) = 100a_1 + 4950d = 6950$. Solving these two equations simultaneously yields $2500d = 2500 \rightarrow d = 1$ which in turn yields $a_1 = \mathbf{20}$.

15. If the tens digit is t and the units digit is u, then we are looking for positive single-digit solutions to $10t + u = 3tu$. By trial and error, when $t = 1,\ u = 5,$ and when $t = 2,$ $u = 4$. When $t > 2$, there are no single-digit values of u. Thus, the larger integer satisfying the given condition is $\mathbf{24}$.

16. Draw $\overline{AD}$. In right ΔADC, $AD = 13$ in the 5-12-13 triangle. Then, since $\overline{DM}$ is a perpendicular bisector of $\overline{AB}$, ΔADB is an isosceles triangle with $AD = BD = 13$. In right ΔABC, using the Pythagorean Theorem, $AB = \sqrt{12^2 + 18^2} = \sqrt{468} = 6\sqrt{13}$ and the required product is $6 \cdot 13 = \mathbf{78}$.



17. First, we can re-write the given equation as $(2x-3)^3 - (2x-3) = 0 \rightarrow$ $(2x-3)[(2x-3)^2 - 1] = 0 \rightarrow (2x-3)(2x-4)(2x-2) = 0 \rightarrow x = 1.5, 2, 1$. The required product is $10 \cdot (1.5 + 2 + 1) = \mathbf{45}$.

18. Re-write the given equation as $\frac{1}{\sin x} + \frac{\cos x}{\sin x} = 3 \rightarrow 1 + \cos x = 3\sin x \rightarrow 1 + \cos x =$ $3\sqrt{1 - (\cos x)^2} \rightarrow (\cos x)^2 + 2\cos x + 1 = 9 - 9(\cos x)^2 \rightarrow 10(\cos x)^2 + 2\cos x - 8 =$ $0 \rightarrow 2(5\cos x - 4)(\cos x + 1) = 0 \rightarrow \cos x = -1$ (reject) or $\cos x = \frac{4}{5} \rightarrow \sin x = \frac{3}{5}$. The required product is then $144\left(\frac{5}{3} - \frac{4}{3}\right) = \mathbf{48}$. Alternatively, multiply both sides of the given equation by $\csc x - \cot x$. Using the identity $(\csc x)^2 - (\cot x)^2 = 1$, $1 = 3(\csc x - \cot x) \rightarrow 144(\csc x - \cot x) = 144\left(\frac{1}{3}\right) = \mathbf{48}$.